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A topology can be generated by $\mathcal{S} \subset \mathcal{P}(X)$ in two steps.

Step 1: Taking finite intersections $\longrightarrow \mathcal{B}$

Step 2: Taking arbitrary unions from $\mathcal{B}$

Examples.

$\mathcal{S} = \{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\} \longrightarrow \mathcal{T}_{std}$

$\mathcal{S} = \{(a,b) : a < b \in \mathbb{R}\} \longrightarrow \mathcal{T}_{std}$

$\cap$

$\mathcal{S} = \{[a,b) : a < b \in \mathbb{R}\} \longrightarrow \mathcal{T}_{ll}$

But, the 2^{nd} & 3^{rd} are actually a base i.e. Step 1 is not needed

Theorem.

$\mathcal{S} \subset \mathcal{P}(X)$ is a base for a topology if

(i) $\phi, X \in \mathcal{S}$

(ii) For each $U, V \in \mathcal{S}$ and $x \in U \cap V$,
$\exists\, W \in \mathcal{S}$, $x \in W \subset U \cap V$

Remark.

The two bases in the example above only satisfy (ii) but not (i).

* This theorem is not if-and-only-if

* Is (i) $\phi, X \in \mathcal{S}$ really essential?

$\phi = \bigcup \phi$ can be created by union,
so $\phi \in \mathcal{S}$ may be omitted
$\{(a,b) : a < b < 100\}$ still satisfies (ii)
but unions at most give $(-\infty, 100) \neq \mathbb{R}$
Thus, $X \in \mathcal{S}$ is needed.

Proof of Theorem.

Our task is just define $\mathcal{J} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{S}\}$ and verify the two rules of topology, i.e., closed under arbitrary union and finite intersection.

For arbitrary union, it is a union of unions.

For finite intersection, let $\bigcup \mathcal{A}_1, \bigcup \mathcal{A}_2 \in \mathcal{J}$.
i.e. $\mathcal{A}_1 = \{P_\alpha : \alpha \in I\}$, $\mathcal{A}_2 = \{Q_\beta : \beta \in J\} \subset \mathcal{B}$

Consider $Z = (\bigcup \mathcal{A}_1) \cap (\bigcup \mathcal{A}_2) = \bigcup_{\alpha,\beta} (P_\alpha \cap Q_\beta)$

For each $x \in Z$, $\exists\, \alpha, \beta$ $x \in P_\alpha \cap Q_\beta$

By assumption (ii) $\exists\, W_x \in \mathcal{S}$ $x \in W_x \subset P_\alpha \cap Q_\beta$

Clearly $Z \subset \bigcup_{x \in Z} W_x \subset \bigcup_{\alpha,\beta} (P_\alpha \cap Q_\beta) = Z$

Thus Z is also in $\mathcal{J}$.

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In the above, we ask if a set $\mathcal{S} \subset \mathcal{P}(X)$ is qualified to make a topology, without knowing which topology.

Another question.

Given a known topology $\mathcal{T}$ and $\mathcal{B} \subset \mathcal{P}(X)$ how do we know if $\mathcal{B}$ is a base for $\mathcal{T}$

Theorem

$\mathcal{B}$ is a base for $\mathcal{T}$, i.e., $\mathcal{T} = \{\cup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$

$\Leftrightarrow \forall\, G \in \mathcal{T}$ and $x \in G$, $\exists\, U \in \mathcal{B}$, $x \in U \subset G$

Quick thought proof

"$\Rightarrow$" For $x \in \cup \mathcal{A} \in \mathcal{T}$, $\exists\, U \in \mathcal{A} \subset \mathcal{B}$ $x \in U$

"$\Leftarrow$" similar to the last step in previous theorem

Definition. Let $x \in X$.

A local base (or nbhd base) at x is a collection $\mathcal{U}_x \subset \mathcal{T}$ such that

$\forall$ nbhd N of x, $\exists\, U \in \mathcal{U}_x$

$x \in U \subset N$

Examples.

* For metric spaces, $\mathcal{U}_x = \{B(x, \frac{1}{n}) : 1 \le n \in \mathbb{Z}\}$
* For $\mathbb{R}, \mathcal{T}_{ll}$, $\mathcal{U}_x = \{[x, x+\varepsilon) : \varepsilon > 0\}$

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Exercise. $\{U_x : x \in X\}$ form a base for $\mathcal{T}$.

Example.

* In the case of metric spaces, at each $x \in X$
  $$\{B(x, \tfrac{1}{k}) : 1 \le k \in \mathbb{Z}\}$$ is countable
* For $X = \mathbb{R}^n$, $\mathcal{T}_{std}$, the situation is better
  $$\{B(q, \tfrac{1}{k}) : 1 \le k \in \mathbb{Z},\ q \in \mathbb{Q}^n\}$$
  is a countable base

Definition. A topological space $(X, \mathcal{T})$ is of

* 1st countable (C_I) if at each $x \in X$, there is a countable local base
* 2nd countable (C_{II}) if there is a countable base for $\mathcal{T}$.